derivation provides an insight into the rotation matrices of Eq. (17). A numerical example is shown to compute q_{bn} in a digital simulation. This derivation can be extended to other coordinate frames also.

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Existence and Uniqueness Proof for the Minimum Model Error Optimal Estimation Algorithm

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Introduction

RECENTLY, a new approach for performing postexperiment optimal state estimation in the presence of significant model error and/or significant measurement error has been developed.[1] The method has been extended and applied to several examples in postexperiment state estimation, system identification, force estimation, and model error determination.[2-6] The new approach, called minimum model error (MME) estimation, avoids some of the theoretical shortcomings of the commonly used Kalman filter-smoother type of algorithms for postexperiment optimal estimation. These topics are discussed at length in Ref. 1. To obtain the MME estimates, a jump-discontinuous two-point boundary-value problem (TPBVP) must be solved. Although excellent results have been obtained using MME as reported in Refs. 1–6, all of this work simply assumed the existence of a solution and then obtained a solution numerically. The purpose of this Note is to provide an existence and uniqueness proof for the solution of the jump-discontinuous TPBVP of the MME.

Necessary Conditions for Minimum Model Error Optimal Estimator

We define the postexperiment optimal estimation problem as follows: Given a system whose state vector dynamics is modeled by the system of equations,

$$\dot{x} = f[x(t), u(t), t]$$

where $x \equiv n \times 1$ state vector, $f \equiv n \times 1$ vector of dynamic model equations, and $u \equiv p \times 1$ vector of forcing terms, and given a set of discrete measurements modeled by the system of equations,

$$\tilde{z}(\tau_j) = g_j[x(\tau_j), \tau_j] + w_j, \qquad j = 1, \ldots, m$$

where $\tilde{z}(\tau_j) \equiv r \times 1$ measurement set at τ_j, $g_j \equiv r \times 1$ measurement model equations, and $w_j \equiv r \times 1$ measurement error vector, determine the optimal estimate for $x(t)$ during some specified time interval $t_0 \le t \le t_f$. The definition of the optimality criterion is typically the distinguishing feature among various optimal estimation strategies. The optimality criterion for the MME is unique and is developed in Ref. 1, along with a discussion of differences between the MME approach and several other approaches.

In the MME, model error is represented by adding a to-be-determined "unmodeled disturbance," vector $d(t)$ to the right-hand sides of the state model equations as

$$\dot{x} = f[x(t), u(t), t] + d(t)$$

Then, the following cost functional is minimized with respect to $d(t)$:

$$J = \sum_{j=1}^{m} \{[\tilde{z}(\tau_j) - g_j(\hat{x}(\tau_j), \tau_j)]^T R_j^{-1} [\tilde{z}(\tau_j) - g_j(\hat{x}(\tau_j), \tau_j)]\} + \int_{t_0}^{t_f} d(\tau)^T W d(\tau)\, \mathrm{d}\tau$$

where $\hat{x}(\tau_j) \equiv n \times 1$ state vector estimate at τ_j and $W \equiv n \times n$ weight matrix. Determination of the matrix W is discussed in Ref. 1.

The necessary conditions for the minimization of J with respect to $d(t)$ follow directly from a modification[7] of the so-called Pontryagin's necessary conditions[8] and lead to the TPBVP summarized as:

$$\dot{x} = f[x(t), u(t), t] + d(t) \tag{1}$$

$$\dot{\lambda} = -\left(\frac{\partial f}{\partial x}\right)^T \lambda \tag{2}$$

$$d = -\frac{1}{2} W^{-1} \left[\frac{\partial f}{\partial u}\right]^T \lambda \tag{3}$$

$$x(t_0) = \text{specified}, \qquad \text{or} \qquad \lambda(t_0^-) = 0 \tag{4}$$

$$\lambda(\tau_j^+) = \lambda(\tau_j^-) + 2H_j^T R_j^{-1} \{\tilde{z}(\tau_j) - g_j[\hat{x}(\tau_j), \tau_j]\} \tag{5}$$

$$x(t_f) = \text{specified}, \qquad \text{or} \qquad \lambda(t_f^+) = 0 \tag{6}$$

where

$$H \equiv \left.\frac{\partial g_j}{\partial x}\right|_{\hat{x}(\tau_j), \tau_j}$$

The existence of a jump discontinuity in the costate vector λ is evident in Eq. (5) at each measurement time τ_j.

Related Work

The MME algorithm requires solution of the jump-discontinuous TPBVP described by Eqs. (1–6). Despite the large volume of work dealing with the continuous TPBVP,[9,10] relatively little work has been performed on the jump-discontinuous problem. This can be partly explained by available methods for converting "special cases," such as the jump-discontinuous problem, into standard form.[11] However, the solution of the transformed problem is often very inefficient.[12] A general-purpose code (PASVA4)[13] has been developed for nonlinear jump-discontinuous TPBVPs and applied to the problem of seismic ray tracing. However, the code is based on a finite-difference approach, requiring that the mesh points coincide with the jump points. For problems of large order or

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containing many jumps, this restriction can be computationally prohibitive.

Multiple Shooting Solution

We now provide mathematical proof of the existence and uniqueness of the MME solution via a multiple shooting approach to the solution of the jump-discontinuous TPBVP. For conciseness of notation in the remainder of this Note, we adopt the following description of the linear TPBVP with jump discontinuities as

$$\mathcal{L}y(t) = \dot{y} - C(t)y = f(t), \qquad t_0 \le t \le t_f, \qquad t \ne \tau_i$$
$$i = 1,2,\ldots,m \quad (7)$$

$$\beta = B_0 y(t_0^-) + B_f y(t_f^+) \quad (8)$$

$$y(\tau_i^+) = D_i y(\tau_i^-) + Y_i, \quad t_0 \le \tau_i \le t_f, \quad i = 1,2,\ldots,m \quad (9)$$

where $y(t)$ is a vector containing both $x(t)$ and $\lambda(t)$; B_0, B_f, and β account for the boundary conditions; and D_i, Y_i account for the jump discontinuities. A multiple shooting approach can be used to convert the jump-discontinuous TPBVP defined by Eqs. (7–9) to a set of simultaneous algebraic equations. The shooting approach is based on conversion to initial-value problems.[14] Assume that a solution exists in the form

$$y(t) = v(t) + V(t)c \quad (10)$$

where $v(t)$ is a particular solution, $V(t)$ an $n \times n$ fundamental solution matrix, and $c(t)$ an n vector of to-be-determined constants. Substituting Eq. (10) into Eq. (8), the boundary conditions may be written as

$$B_0[v(t_0) + c(t)] + B_f[v(t_f) + V(t_f)c] = \beta$$

which may be rearranged to obtain the solution for c,

$$c = [B_0 + B_f V(t_f)]^{-1}[\beta - B_0 v(t_0) - B_f v(t_f)]$$

assuming that the inverse exists. The quantities $V(t_f)$ and $v(t_f)$ are obtained by integrating

$$\mathcal{L}v(t) = f(t), \qquad v(t_0) = v_0$$

$$\mathcal{L}V(t) = 0, \qquad V(t_0) = I$$

where v_0 is specified. Difficulties of simple shooting usually arise due to stiffness in the differential equations.[15] Multiple shooting is used to alleviate stiffness problems. The domain is divided into subintervals as

$$t_0 < t_1 < t_2 < \cdots < t_N = t_f$$

and then simple shooting is used in each subinterval, with continuity conditions imposed on the solution at the nodes t_j. A good discussion of multiple shooting for continuous problems is contained in Ref. 16. Here, we proceed directly to the jump-discontinuous problem. First, consider the case that the shooting nodes t_j coincide with the jump times τ_j. The solution $y(t)$ can be expressed in the interval $t_{j-1} < t < t_j$ as

$$y(t) = y_j(t) = v_j(t) + V_j(t)c_j, \qquad t_{j-1} < t < t_j \quad (11)$$

where we require that

$$\mathcal{L}v_j(t) = f(t), \qquad v_j(t_{j-1}^+) = v_j^0, \qquad t_{j-1} < t < t_j \quad (12)$$

$$\mathcal{L}V_j(t) = 0, \qquad V_j(t_{j-1}^+) = V_j^0, \qquad t_{j-1} < t < t_j \quad (13)$$

The continuity conditions must be modified to account for jump discontinuities. At t_j^-, the solution is represented by

$$y(t_j^-) = y_j(t_j^-) = v_j(t_j^-) + V_j(t_j^-)c_j \quad (14)$$

Substituting Eq. (14) into Eq. (9) yields

$$y(t_j^+) = D_j[v_j(t_j^-) + V_j(t_j^-)c_j] + Y_j \quad (15)$$

From Eqs. (11–13), we may also write

$$y(t_j^+) = y_{j+1}(t_j^+) = v_{j+1}^0 + V_{j+1}^0 c_{j+1} \quad (16)$$

Comparing Eqs. (15) and (16), we require that

$$D_j[v_j(t_j^-) + V_j(t_j^-)c_j] + Y_j = v_{j+1}^0 + V_{j+1}^0 c_{j+1} \quad (17)$$

The condition [Eq. (17)] is imposed at each jump point.

We now evaluate the boundary condition, Eq. (8). At the initial time, assuming a jump exists at t_0, we have

$$y(t_0^-) = v_0^0 + V_0^0 c_0 \quad (18)$$

$$y(t_0^+) = D_0[v_0^0 + V_0^0 c_0] + Y_0 \quad (19)$$

From Eqs. (16) and (17), we also have

$$y(t_0^+) = v_1^0 + V_1^0 c_1 \quad (20)$$

and

$$D_0[v_0(t_0^-) + V_0(t_0^-)c_0] + Y_0 = v_1^0 + V_1^0 c_1 \quad (21)$$

At the final time, assuming it is a jump point, we have

$$y(t_f^+) = D_k[v_k(t_f^-) + V_k(t_f^-)c_k] + Y_k \quad (22)$$

Substituting Eqs. (18) and (22) into Eq. (8), the boundary conditions may be written as

$$B_0(v_0^0 + V_0^0 c_0) + B_f\{D_k[v_k(t_f^-) + V_k(t_f^-)c_k] + Y_k\} = \beta \quad (23)$$

Combining Eqs. (16–23), we obtain

$$\begin{bmatrix} B_0 V_0^0 & 0 & 0 & \cdots & B_f D_k V_k(t_f^-) \\ -D_0 V_0^0 & V_1^0 & 0 & \cdots & 0 \\ 0 & -D_1 V_1(t_j^-) & V_2^0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & -D_{k-1}V_{k-1}(t_{k-1}^-) & V_k^0 \end{bmatrix}$$

$$\times \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix} = \begin{bmatrix} \beta - B_0 v_0^0 - B_f D_k v_k(t_f^-) - B_f Y_k \\ D_0 v_0^0 - v_1^0 + Y_0 \\ D_1 v_1(t_1^-) - v_2^0 + Y_1 \\ \vdots \\ D_{k-1} v_{k-1}(t_{k-1}^-) - v_k^0 + Y_{k-1} \end{bmatrix} \quad (24)$$

Equation (24) represents a system of $2n * (k+1)$ equations for the $k+1$ unknown $2n$ vectors $c_0, c_1, \ldots, c_k$. From a practical standpoint, if the number of jump discontinuities is large, computational difficulties may arise if the shooting nodes coincide with every jump point. To alleviate this situation, the solution may be modified to allow multiple jump points to be present in a single subinterval. For example, we show $m+1$ jumps in the jth subinterval as

$$t_{j-1} \le \tau_i \le \tau_{i+1} < \cdots < \tau_{i+m} \le t_j$$

For simplicity of presentation, consider the case when the nodes coincide with every second jump point, represented by

$$t_{j-1} = \tau_i < \tau_{i+1} < \tau_{i+2} = t_j$$

As in simple shooting, in the subinterval $t_{j-1} < t < t_j$, $t \neq \tau_{i+1}$, we write

$$y(t) = y_j(t) = v_j(t) + V_j(t)c_j, \qquad t_{j-1} < t < t_j$$

$$v_j(t_{j-1}^+) = v_j^0, \qquad V_j(t_{j-1}^+) = V_j^0$$

At the jump point τ_{i+1}, we may write

$$y(\tau_{i+1}^+) = D_{i+1}y(\tau_{i+1}^-) + Y_{i+1}$$

$$= D_{i+1}[v_j(\tau_{i+1}^-) + V_j(\tau_{i+1}^-)c_j] + Y_{i+1}$$

In the subinterval $t_{j-1} < t < t_j$, we require

$$\mathcal{L}V_j(t) = 0, \qquad V_j(t_{j-1}^+) = V_j^0, \qquad t \neq \tau_{i+1}$$

$$V_j(\tau_{i+1}^+) = D_{i+1}V_j(\tau_{i+1}^-)$$

$$\mathcal{L}v_j(t) = f(t), \qquad v_j(t_{j-1}^+) = v_j^0, \qquad t \neq \tau_{i+1}$$

$$v_j(\tau_{i+1}^+) = D_{i+1}v_j(\tau_{i+1}^-)$$

The jump at time τ_{i+2} is given by

$$y(\tau_{i+2}^+) = D_{i+2}y(\tau_{i+2}^-) + Y_{i+2}$$

$$= D_{i+2}[v_j(\tau_{i+2}^-) + V_j(\tau_{i+2}^-)c_j] + Y_{i+2} \tag{25}$$

Since $t_j = \tau_{i+2}$, the "initial condition" for the $(j + 1)$th subinterval is

$$y(\tau_{i+2}^+) = v_{j+1}^0 + V_{j+1}^0 c_{j+1} \tag{26}$$

Equations (25) and (26) may be combined to yield

$$-D_{i+2}V_j(\tau_{i+2}^-)c_j + V_{j+1}^0 c_{j+1} = D_{i+2}v_j(\tau_{i+2}^-) - v_{j+1}^0 + Y_{i+2}$$

In general, an arbitrary number of jumps may be combined into a single subinterval. Furthermore, the node points for the subinterval do not need to coincide with any of the jump points. A set of linear algebraic equations similar to those shown in Eq. (24) may be constructed for the general problem in which the number of jumps contained in each subinterval may vary and the node points do not necessarily coincide with the jump points.

Proof of the Existence and Uniqueness of the Minimum Model Error Solution

Theorem. If $A(t)$ and $f(t)$ are continuous in $[t_0, t_f]$, the problem defined by Eqs. (7-9) has a unique solution if and only if $B_0 + B_f V_f$ is nonsingular.

Proof. Without loss of generality, let $t_0 = \tau_0$ and $t_f = \tau_k$. At the initial time, we may write

$$y(\tau_0^+) = v_0(\tau_0^-) + V_0(\tau_0^-)c \tag{27}$$

Here $v_0(\tau_0^-) = 0$, $V_0(\tau_0^-) = I_{n \times n}$. Consider the initial state jump. From Eqs. (9) and (27), we write

$$y(\tau_0^+) = D_0V_0(\tau_0^-)c + [D_0v_0(\tau_0^-) + Y_0] \tag{28}$$

In the first interval, define

$$\mathcal{L}V_1(t) = 0, \qquad V_1(\tau_0^+) = D_0V_0(\tau_0^-), \qquad t_0 < t < \tau_1 \tag{29}$$

$$\mathcal{L}v_1(t) = f(t), \qquad v_1(\tau_0^+) = D_0v_0(\tau_0^-) + Y_0, \qquad t_0 < t < \tau_1 \tag{30}$$

The jump discontinuity at τ_1 is described by

$$y(\tau_1^+) = D_1V_1(\tau_1^-)c + [D_1v_1(\tau_1^-) + Y_1]$$

Repeating the procedure [Eqs. (28-30)] throughout the domain, we write

$$\mathcal{L}V_2(t) = 0, \qquad V_2(\tau_1^+) = D_1V_1(\tau_1^-), \qquad \tau_1 < t < \tau_2$$

$$\mathcal{L}v_2(t) = f(t), \qquad v_2(\tau_1^+) = D_1v_1(\tau_1^-) + Y_1, \qquad \tau_1 < t < \tau_2$$

$$\vdots \qquad \vdots$$

$$\mathcal{L}V_{j+1}(t) = 0, \qquad V_{j+1}(\tau_j^+) = D_jV_j(\tau_j^-), \qquad \tau_j < t < \tau_{j+1}$$

$$\mathcal{L}v_{j+1}(t) = f(t), \qquad v_{j+1}(\tau_j^+) = D_jv_j(\tau_j^-) + Y_j, \qquad \tau_j < t < \tau_{j+1}$$

$$\vdots \qquad \vdots$$

$$\mathcal{L}V_k(t) = 0, \qquad V_k(\tau_{k-1}^+) = D_{k-1}V_{k-1}(\tau_{k-1}^-), \qquad \tau_{k-1} < t < \tau_k = t_f$$

$$\mathcal{L}v_k(t) = f(t), \; v_k(\tau_{k-1}^+) = D_{k-1}v_{k-1}(\tau_{k-1}^-) + Y_k, \; \tau_{k-1} < t < \tau_k = t_f$$

At the final jump, we have

$$y(t_f^+) = D_kV_k(t_f^-)c + [D_kv_k(t_f^-) + Y_k]$$

Let

$$V_f = D_kV_k(t_f^-)$$

$$v_f = D_kv_k(t_f^-) + Y_k$$

The boundary condition can be written as

$$(B_0 + B_fV_f)c = \beta - B_fv_f \tag{31}$$

From Eq. (31), we see that c has a unique solution if, and only if, $(B_0 + B_fV_f)$ is nonsingular.

Q.E.D.

Conclusion

The minimum model error (MME) optimal estimation algorithm requires solution of a jump-discontinuous two-point boundary-value problem (TPBVP). Although considerable previous work has been devoted to the solution of continuous TPBVPs, relatively little work has been devoted to the jump-discontinuous TPBVP. In this Note, a general solution to the jump-discontinuous TPBVP of the MME is presented and its existence and uniqueness are proved.

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Compensating Sampling Errors in Stabilizing Helmet-Mounted Displays Using Auxiliary Acceleration Measurements

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Introduction

THE visual acuity in helmet-mounted displays (HMD) under conditions of aircraft vibrations may be imparied by uncontrollable vibratory relative motion between the viewer's eye point-of-regard and the display. This vibratory motion often causes image blurring because the HMD vibrates with the pilot's head while his eyes are stabilized inertially by the vertibulo-ocular reflex.[1] To retain visual acuity, image stabilization on the HMD is needed.

Previous studies of Wells and Griffin[2,3] have shown the potential benefits of image stabilization in HMDs for retaining visual acuity under the conditions of whole-body vibrations. In their studies, the stabilization was performed by utilizing the double integrated head rotational acceleration signal, derived from at least one pair of helmet-mounted accelermeters, to shift the image on the display. This method, however, performed poorly when large-amplitude head rotations were executed.

An alternative method, which makes use of adaptive filtering to estimate the head motion due to the platform accelerations and uses this estimate to shift the image on the display, was devised.[4,5] This estimate is based on the readings of the helmet position and orientation measurement device. Such a device, whether magnetic or electro-optic, typically has a sampling rate of 30–60 Hz, which is dictated by computation time, sensor response time, and noise.[6] Futhermore, signal transmission and display computation time further reduce the effective rate. Head motion measurement devices with substantially higher rates are presently unavailable.

Since the desired image stabilization is in the frequency range of 3–8 Hz, the delay due to the sampling rate introduces a large phase shift, resulting in a significant stabilization error. This can be demonstrated by the following example. Denote θ as the true head orientation; thus, the delayed measurement would be $\theta\epsilon^{-\tau s}$ and the relative error can be expressed by

$$\epsilon = (\theta - \theta e^{-\tau s})/\theta = 1 - e^{-\tau s} = 1 - e^{-j\omega\tau} = 1 - \cos\omega\tau + j\sin\omega\tau \qquad (1)$$

The absolute value of ϵ is

$$|\epsilon| = \left[(1-\cos\omega\tau)^2 + \sin^2\omega\tau\right]^{1/2} = \sqrt{2}(1-\cos\omega\tau)^{1/2} \qquad (2)$$

For example, with $f = 5$ Hz and $\tau = 30$ ms, $|\epsilon| = 0.91$, i.e., 91% error. To reduce this error to an acceptable level of 10%, the sampling rate must be increased to about 300 Hz.

A method that overcomes this obstacle, based on complementary filtering, is described. It combines measurements of the head position and orientation system with measurements of the angular accelerations of the head, so as to reduce the phase shift in the head orientation measurements. Moreover, the complementary filter, if implemented either as an analog circuit, or digitally at much higher frequencies, substantially improves the resolution of the output signal.

Complementary Filter

A block diagram of the complementary filter is shown in Fig. 1. The two inputs to the filter are the head orientation vector $\boldsymbol{\Theta}$ and the head angular acceleration vector $\ddot{\boldsymbol{\Theta}}$. For brevity, only one component of each vector will be addressed in the sequel, i.e., θ and $\ddot{\theta}$, respectively. The orientation signal θ is obtained from the measurement system as a digital stream with a frequency of 30 Hz. As a result of the computations in the measurement system, this signal is delayed by T s and is constant for the duration of the sampling interval. The acceleration is measured at a much higher sampling rate and can be considered as a continuous signal. Referring to Fig. 1, the equation of the complementary filter is

$$\left\{\left[(1-e^{-Ts}/Ts)e^{-Ts}\theta - \hat{\theta}\right]\frac{a_0}{s} + s^2\theta + b + \left[(1-e^{-Ts}/Ts)e^{-Ts}\theta - \hat{\theta}\right]a_1\right\}\frac{1}{s} + \left[(1-e^{-Ts}/Ts)e^{-Ts}\theta - \hat{\theta}\right] \times a_2 = s\hat{\theta} \qquad (3)$$

The output of the filter is the estimated head orientation, and accordingly is

$$\hat{\theta} = \frac{s^3 + a_2\eta s^2 + a_1\eta s + a_0\eta}{s^3 + a_2 s^2 + a_1 s + a_0}\,\theta + \frac{s}{s^3 + a_2 s^2 + a_1 s + a_0}\,b \qquad (4)$$

where

$$\eta = \left[(1-e^{-Ts})/Ts\right]e^{-Ts}$$

and b is the bias of the accelerometer or noise. From Eq. (4) it is apparent that at high frequencies $\hat{\theta}\to\theta$. At low frequencies $\eta\to 1$, so that again $\hat{\theta}\to\theta$. The same argument holds for high sampling rates when $T\to 0$ again, $\eta\to 1$, and $\hat{\theta}\to\theta$. Also, from Eq. (4) it is clear that both the low- and high-frequency accel-

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